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Initial motion of a bubble in a fluidized bed

Part 1. Theory

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This is a theoretical study of the experimental work in the succeeding paper by Partridge & Lyall (1967). This paper shows that the characteristic shape of typical bubbles in fluidized beds, with a large solids to fluid density ratio, will be reached after a time of order $(r_0/g)^{\frac{1}{2}}$ from the introduction, naturally or artificially, into the bed of a circular or spherical bubble, where r_0 is the initial bubble radius and g the gravitation constant. The initial acceleration is g in two dimensions and $2g$ in three dimensions. A measure of the growth rate of the distortion from a circular or spherical shape is given as a tentative guide to wake formation and growth rate. It is shown that the wake growth rate in the three-dimensional case is faster than in the two-dimensional case, and, further, that bubbles formed naturally at the bottom of a fluidized bed will distort faster than bubbles starting from rest.

The method used is based on a convective term linearization process on equations of motion for a fluidized system given by Murray (1965*a*, *b*) and an extension of a method given by Walters & Davidson (1962, 1963) in the case of the initial motion of a gas bubble formed in an inviscid liquid and starting from rest.

Comparison with experiment is difficult since the bubble conditions discussed in the theory are very difficult to reproduce artificially in the laboratory. The results obtained by Partridge & Lyall (1967) are shown to be not inconsistent with those predicted here.

1. Introduction

Bubbles, or voids of particles, appear in most gas-fluidized beds. Empirically bubbles occur in beds where $\rho_s/\rho_f > 10$, where ρ_s, ρ_f are the respective solids and fluid densities, and so the study of bubbles is not solely restricted to gas beds. Bubbles are the prime cause of the good particle-mixing properties of gas-fluidized beds as shown by Rowe & Partridge (1962).

The wake of particles which moves with the bubble is important in particle mixing and it seems of interest, therefore, to study the initial rate of distortion from a circular or spherical void introduced into a bed in an attempt to obtain information on the growth of the wake. It is also of interest in the general study of bubble motion in fluidized beds.

This investigation is theoretical and the results compared with known experi-

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mental facts and with the experimental work in the succeeding paper by Partridge & Lyall (1967). Fair comparison is found with the experimentally measureable quantities. The equations used below are those suggested by Murray (1965*a*). It is shown that the dimensionless particle concentration, Z, may be reasonably approximated by a constant in this unsteady case in a similar manner to that given by Murray (1965*b*), who studied the steady motion of fully developed bubbles. He also discussed a possible steady wake configuration. The subsequent equations with Z constant are shown to admit of potential flows for the solids and gas motions when the convective terms are linearized using a modified Oseen technique. The method given by Walters & Davidson (1962, 1963) can then be extended to fluidized-bed bubbles. Their method deals with the initial motion from rest of a gas bubble formed in an inviscid liquid. They assume that potential flow will persist for the initial motion, that the pressure is constant inside the bubble, and that distortions from the initial given shape are small for small times.

In this paper the situation is a two-phase flow and unlike Walters & Davidson's case is reduced to a linear problem. This linearity is a necessity for the consistency of the constant Z solution and the subsequent irrotationality.

2. Equations of motion and their subsequent linear form

The continuum equations of motion used are the 'inviscid'† equations derived by Murray (1965*a*), which are, for fluidized beds with $\rho_s/\rho_f \gg 1$,

$$(\partial Z/\partial t') + \operatorname{div} Z\mathbf{v}_s' = 0, \tag{1}$$

$$-(\partial Z/\partial t') + \operatorname{div}(1-Z)\,\mathbf{v}_f' = 0, \tag{2}$$

$$\rho_s Z[(\partial \mathbf{v}_s'/\partial t') + (\mathbf{v}_s'.\operatorname{grad})\,\mathbf{v}_s'] = -g\rho_s Z\mathbf{i} + D'(Z)\,(\mathbf{v}_f' - \mathbf{v}_s'), \tag{3}$$

$$\operatorname{grad} p' = -D'(Z)\,(\mathbf{v}_f' - \mathbf{v}_s'), \tag{4}$$

where, in obtaining (1)–(4), ρ_s, ρ_f are taken to be constant, Z is the dimensionless particles volume fraction, $\mathbf{v}_f'$, $\mathbf{v}_s'$ are the continuum fluid and solids velocities, p' is the pressure in the fluid, $D'(Z)$ is the total drag on the particles (defined by Murray 1965*a*), t' is the time and $\mathbf{i}$ is the unit vector in the vertical x'-direction.

The following non-dimensional quantities are introduced, where v_0 is the interstitial fluid velocity far from any bubble, r_0 the initial bubble radius which is introduced into the bed at $t' = 0$, $U'(t')$ the velocity of the bubble at time t', and $\mathbf{r}'$ the space vector:

$$\left.\begin{aligned} \mathbf{v}_f' = v_0\mathbf{v}_f, \quad \mathbf{v}_s' = v_0\mathbf{v}_s, \quad t' = r_0 t/v_0,\\ (x',y',z') = \mathbf{r}' = r_0\mathbf{r}, \quad D'(Z) = \rho_s v_0 D(Z)/r_0,\\ p' = \rho_s v_0^2 p, \quad F = v_0^2/gr_0, \quad U' = v_0 U(t),\\ T = t/F^{\frac{1}{2}} = t'(g/r_0)^{\frac{1}{2}}, \quad V = U(0)F^{\frac{1}{2}} = U'(0)/(gr_0)^{\frac{1}{2}}. \end{aligned}\right\} \tag{5}$$

Using (5), equations (1)–(4) become in dimensionless form

$$(\partial Z/\partial t) + \operatorname{div} Z\mathbf{v}_s = 0, \tag{6}$$

$$-(\partial Z/\partial t) + \operatorname{div}(1-Z)\,\mathbf{v}_f = 0, \tag{7}$$

$$Z[(\partial \mathbf{v}_s/\partial t) + (\mathbf{v}_s.\operatorname{grad})\,\mathbf{v}_s] = -\mathbf{i}Z/F + D(Z)\,(\mathbf{v}_f - \mathbf{v}_s), \tag{8}$$

$$\operatorname{grad} p = -D(Z)\,(\mathbf{v}_f - \mathbf{v}_s). \tag{9}$$

† The continuum solids and gas stress tensors are omitted (see Murray 1965*a*).

The origin of the co-ordinates is at the centre of the bubble $\mathbf{r} = 0$, which rises, in the $\mathbf{i}$-direction, with a velocity $U(t)$. Motion far from the bubble, where $Z = Z_0$, is given by (8), (9) as

$$\left.\begin{aligned} \mathbf{v}_f = \mathbf{i}, \quad \mathbf{v}_s = 0, \quad Z = Z_0, \\ p = p_0 - \frac{Z_0}{F}\left[x + \int_0^t U(\tau)\,d\tau\right]. \end{aligned}\right\} \tag{10}$$

Equations (6)–(9) are now perturbed about the uniform state (10) (far from the bubble). The perturbed quantities are here denoted by asterisks, and from (6)–(10) satisfy

$$\begin{aligned} (\partial Z^*/\partial t) + Z_0 \operatorname{div} \mathbf{v}_s^* &= 0, \\ -(\partial Z^*/\partial t) - (\partial Z^*/\partial x) + (1 - Z_0)\operatorname{div} \mathbf{v}_f^* &= 0, \\ Z_0(\partial \mathbf{v}_s^*/\partial t) &= -\mathbf{i}Z^*/F + D(Z_0)(\mathbf{v}_f^* - \mathbf{v}_s^*) + \mathbf{i}Z^*(dD/dZ)_{Z_0}, \\ \operatorname{grad} p^* &= -D(Z_0)(\mathbf{v}_f^* - \mathbf{v}_s^*) - \mathbf{i}Z^*(dD/dZ)_{Z_0}, \end{aligned}$$

the first three of which give a single equation for Z^*, namely

$$\frac{\partial^2 Z^*}{\partial t^2} + \frac{1}{F(1-Z_0)}\frac{\partial Z^*}{\partial t} + \left[\left(\frac{dD}{dZ}\right)_{Z_0} - \frac{(1-2Z_0)}{F(1-Z_0)}\right]\frac{\partial Z^*}{\partial x} = 0.$$

The only solution of this equation which tends to zero as $r \to \infty$ is $Z^* \equiv 0$. Thus $Z = Z_0$ is the solution for the linearized set of equations above.

If we now introduce a real function $c(t)$† (see also §4) by writing in (8)

$$c(t)\partial \mathbf{v}_s/\partial x \quad \text{for} \quad (\mathbf{v}_s.\operatorname{grad})\mathbf{v}_s \tag{11}$$

in an extension of the modified Oseen process introduced by Lewis & Carrier (1949) and used by Murray (1965*b*), where here the undetermined $c(t)$ is a function of t, the solution for Z^* above is unchanged. We now approximate to the actual solution by taking $Z = Z_0$ throughout the whole field of flow and get, with the above linearization (11) of the convective terms, the following linear set of equations for $\mathbf{v}_f$, $\mathbf{v}_s$, p:

$$\operatorname{div} \mathbf{v}_s = 0 = \operatorname{div} \mathbf{v}_f, \tag{12}$$

$$(\partial \mathbf{v}_s/\partial t) + c(t)(\partial \mathbf{v}_s/\partial x) = -\mathbf{i}/F + (\mathbf{v}_f - \mathbf{v}_s)/F, \tag{13}$$

$$\operatorname{grad} p = -(Z_0/F)(\mathbf{v}_f - \mathbf{v}_s), \tag{14}$$

only three of which are independent ((13), (14), and one of (12)). With $\mathbf{v}_f$, $\mathbf{v}_s$ irrotational in the uniform state far from the bubble, (12)–(14) show that $\mathbf{v}_f$, $\mathbf{v}_s$ are irrotational everywhere, and that p is harmonic. As shown by Murray (1965*b*), the irrotationality and the $Z = Z_0$ solution are a consequence of the linearization process, and so the non-linear form $(\mathbf{v}_s.\operatorname{grad})\mathbf{v}_s$ cannot be used in (13). Equations (13) and (14) thus give a Bernoulli-type equation with $\frac{1}{2}\mathbf{v}_s^2$ linearized to $c(t)\,v_s$.

Disturbance quantities (not small) are now introduced with the use of primes (*not* to be confused with those for dimensional quantities above)

$$\left.\begin{aligned} \mathbf{v}_f = \mathbf{i} + \mathbf{v}_f', \quad \mathbf{v}_s = \mathbf{v}_s', \\ p = p_0 - \frac{Z_0}{F}\left[x + \int_0^t U(\tau)\,d\tau\right] + p', \end{aligned}\right\} \tag{15}$$

† Its inclusion allows more freedom in obtaining a better approximate solution.

where $\mathbf{v}_f'$, $\mathbf{v}_s'$, p' all tend to zero as $r \to \infty$. $\mathbf{v}_f$, $\mathbf{v}_s$, $\operatorname{grad} p$ can now be expressed in terms of derivatives of harmonic functions.

The case of a two-dimensional cylindrical bubble introduced into the bed will be discussed in detail in §§3–7 and some corresponding results given in §8 for a three-dimensional bubble. Details of the latter case are given in the appendix.

3. Method and solution for the two-dimensional case

Introduce the complex potentials $w_f(z,t)$, $w_s(z,t)$ for disturbance quantities $\mathbf{v}_f'$, $\mathbf{v}_s'$ and the complex harmonic function $P(z,t)$, where $p' = \operatorname{Re} P(z,t)$. Equations (13) and (14) give, using (15),

$$\left.\begin{aligned}(\partial w_s/\partial t) + c(t)\,(\partial w_s/\partial z) - F^{-1}(w_f - w_s) &= 0,\\ P + (Z_0/F)\,(w_f - w_s) &= 0.\end{aligned}\right\} \tag{16}$$

We now write w_f, w_s, P as harmonic series, which tend to zero at infinity, in the form

$$\left.\begin{aligned}w_s &= -U(t)/z + \sum_{m=2}^{\infty} s_m(t)/z^m = \sum_{m=1}^{\infty} s_m(t)/z^m,\\ w_f &= \sum_{m=1}^{\infty} f_m(t)/z^m, \quad P = \sum_{m=1}^{\infty} p_m(t)/z^m,\end{aligned}\right\} \tag{17}$$

where the functions of time s_m, f_m, p_m must be chosen so that the pressure p in (15) is constant inside the bubble. This is the essence of the method given by Walters & Davidson (1962, 1963), who, in the equivalent of (17), use polar co-ordinates and the scalar potential for the single velocity. In their case p is not harmonic.

Substitution of (17) into (16), noting that

$$\frac{\partial w_s}{\partial t} = \sum_{m=1}^{\infty} (\dot{s}_m/z^m + mUs_m/z^{m+1}),$$

since the origin of the co-ordinate system is moving upwards with velocity U, gives the following differential recurrence relations on equating powers of z:

$$\left.\begin{aligned}\dot{s}_m - F^{-1}(f_m - s_m) &= -(m-1)\,[U - c(t)]\,s_{m-1},\\ p_m &= -(Z_0/F)\,(f_m - s_m) \quad (m \geqslant 1).\end{aligned}\right\} \tag{18}$$

The constant pressure condition inside the bubble is used in a similar manner to that given by Walters & Davidson (1962) and is as follows. For small times we anticipate small changes in bubble shape from $r = 1$ at $t = 0$ to $r = r_b(\theta, t)$, say, for $t > 0$, and we introduce $\xi(\theta, t)$ by

$$r_b(\theta, t) = 1 + \xi(\theta, t), \tag{19}$$

where ξ will be small for small t. If $p = p_b$ inside the bubble the second of (15), (17) and (19), and the constant pressure condition require

$$p_b - p_0 = \frac{Z_0}{F}\left[(1+\xi)\cos\theta + \int_0^t U(\tau)\,d\tau\right] + \mathrm{Re}\left[\sum_{m-1}^{\infty} p_m/z^m\right]_{r=1+\xi},$$

$$= \frac{Z_0}{F}\left[(1+\xi)\cos\theta + \int_0^t U(\tau)\,d\tau\right] + \sum_{m=1}^{\infty} p_m \cos m\theta$$
$$\times\left[1 - m\xi + \tfrac{1}{2}m(m+1)\,\xi^2 + O(\xi^3)\right],$$

$$= \left[-\frac{Z_0}{F}\left\{\cos\theta + \int_0^t U(\tau)\,d\tau\right\} + \sum_{m=1}^{\infty} p_m \cos m\theta\right]$$
$$-\xi\left[\frac{Z_0}{F}\cos\theta + \sum_{m=1}^{\infty} m p_m \cos m\theta\right] + O(\xi^2),$$

to be independent of θ. This can be done to $O(1)$ if the p_m are such that

$$p_m = Z_0\,\delta_{m1}/F, \tag{20}$$

where δ_{m1} is the Kronecker delta. If we write $U_0 = U(0)$, (18) and (20) now give

$$\left.\begin{aligned} s_m &= -(U_0 + t/F)\,\delta_{m1} - (m-1)\int_0^t [U(\tau) - c(\tau)]\,s_{m-1}\,d\tau,\\ f_m &= s_m - \delta_{m1}. \end{aligned}\right\} \tag{21}$$

From (21),

$$\left.\begin{aligned} U(t) = -s_1(t) &= U_0 + t/F,\\ f_1(t) &= -(1 + U_0 + t/F). \end{aligned}\right\} \tag{22}$$

When $c(t)$ has been determined, (21) gives s_m, f_m and the unsteady flow patterns can be obtained from (15) and (17). Various forms for $c(t)$ are derived in §4.

In view of the linearity we can write $c(t)$ as

$$c(t) = \sum_{l=0}^{\infty} c_l\,t^l, \tag{23}$$

where the c_l are constants which are evaluated in §4. The general form for s_m and f_m can then be found in terms of the c_l. The c_l are chosen so that the resulting solutions will satisfy, as closely as possible, some average condition on the non-linear equation.

With $U(t)$ from (22) and $c(t)$ from (23), it is seen from (21) that the appropriate expansion for s_m for small times is

$$s_m = t^{m-1}\sum_{l=0}^{\infty} {}_m\sigma_l\,t^l, \tag{24}$$

which on substitution in (21) gives the recurrence relation

$${}_m\sigma_l = [(m-1)/(m-1+l)]\,[{}_{m-1}\sigma_l(c_0 - U_0) + {}_{m-1}\sigma_{l-1}(c_1 - F^{-1})$$
$$+ {}_{m-1}\sigma_{l-2}\,c_2 + \dots + {}_{m-1}\sigma_0\,c_l]. \tag{25}$$

f_m is now given by (21). With the c_l from §4 the solutions are given explicitly.

4. Evaluation of $c(t)$

Various methods can be used to obtain a form for $c(t)$. All of those suggested here assign a form to $c(t)$ such that the resulting solutions will satisfy some averaging condition involving the non-linear convective terms but with $Z = Z_0$. As shown below the actual character of the solution is fairly insensitive to the method used, as it should be, of course. The inclusion of $c(t)$ is a device which allows slightly greater freedom in the linear theory to approximate the exact solution.

In boundary-layer-type problems for which the method was introduced, where $c(t)$ is constant, by Lewis & Carrier (1949) it is clear physically what should be done. Here the problem does not have a boundary-layer character and so their reasoning is not directly applicable. The following methods, however, are extensions suggested from their discussion.

It seems reasonable to require the non-linear solids equation (with $Z = Z_0$) for the disturbance quantities to be satisfied on the average. That is, we require

$$\int_1^\infty \frac{\partial}{\partial z}\left[\frac{\partial w_s}{\partial t}+\frac{1}{2}\left(\frac{\partial w_s}{\partial z}\right)^2-\frac{1}{F}(w_f-w_s)\right]dz = 0. \tag{26}$$

Alternatively the first moment of the integrand in (26) could be used instead, giving

$$\int_1^\infty z\frac{\partial}{\partial z}\left[\frac{\partial w_s}{\partial t}+\frac{1}{2}\left(\frac{\partial w_s}{\partial z}\right)^2-\frac{1}{F}(w_f-w_s)\right]dz = 0. \tag{27}$$

Since it was the non-linear convective term which was approximated it is also reasonable to require, say, the integral of the difference of the exact and approximate convective terms to be zero. Suitable weighting, with the velocity, for example, of this difference could also be used in the integral. This results in $c(t)$ being given by integrals of the form

$$\int_1^\infty \left(\frac{\partial w_s}{\partial z}\right)^\alpha \frac{\partial}{\partial z}\left[\frac{1}{2}\left(\frac{\partial w_s}{\partial z}\right)^2-c(t)\frac{\partial w_s}{\partial z}\right]dz = 0, \tag{28}$$

where α is an integer.

With $c(t)$ given by (26), (27) or (28) it is anticipated that the resulting w_f, w_s, P will approximate reasonably at least the gross qualitative features of the exact solution.

Using thes econd of (21), (24) and (25), equations (26), (27) and (28) give respectively

$$c(t) = -\frac{1}{2}\sum_{m=1}^\infty ms_m, \tag{29}$$

$$c(t) = -\sum_{m=1}^\infty \frac{m+3}{m+2}[ms_ms_1+(m-1)s_{m-1}2s_2+\ldots+s_1ms_m]\Big/2\sum_{m=1}^\infty(m+1)s_m, \tag{30}$$

$$c(t) = -\left(\frac{\alpha+1}{\alpha+2}\right)\sum_{m=1}^\infty ms_m. \tag{31}$$

Equation (29) is the same as the special case, $\alpha = 0$, of (31).

With s_m from (24) with (25), (29)–(31) determine the solution, for each $c(t)$ used, in terms of U_0 and F. All of (29)–(31) determine the c_l in terms of the ${}_m\sigma_l$, which on substitution in (25) give recurrence relations in terms of ${}_m\sigma_l$ alone.

5. Initial change in bubble shape and estimate of wake growth rate: $U_0 \neq 0$

A first approximation to the actual bubble shape for small times is obtained by equating the normal solids velocity at $r = r_b$ with the changing shape. Thus, we write

$$\dot{r}_b + U \cos\theta = [\partial \operatorname{Re} w_s / \partial r]_{r=r_b=1+\xi}.$$

From (17) and (19), with ξ small, this gives the equation for ξ as

$$\dot{\xi} - \xi \sum_{m=1}^{\infty} m(m+1) s_m \cos m\theta = -U\cos\theta - \sum_{m=1}^{\infty} m s_m \cos m\theta + O(\xi^2).$$

Thus, to $O(\xi)$,

$$\dot{\xi} - \xi \sum_{m=1}^{\infty} m(m+1) s_m \cos m\theta = -\sum_{m=2}^{\infty} m s_m \cos m\theta. \tag{32}$$

With s_m from (24), it is appropriate to write

$$\xi(\theta, t) = t^2 \sum_{l=0}^{\infty} \xi_l t^l, \tag{33}$$

which on substitution in (32) gives

$$\left.\begin{aligned} \xi_l &= \frac{1}{l+2}[\xi_{l-1} M_0 + \xi_{l-2} M_1 + \ldots + \xi_0 M_{l-1} - N_l] \quad (l \leqslant 2) \\ \text{with}\quad M_l &= 2\,{}_1\sigma_l \cos\theta + 2\cdot 3\,{}_2\sigma_{l-1}\cos 2\theta + \ldots + (l+1)(l+2)\,{}_{l+1}\sigma_0 \cos(l+1)\theta, \\ N_l &= 2\,{}_2\sigma_l \cos 2\theta + 3\,{}_3\sigma_{l-1}\cos 3\theta + \ldots + (l+2)\,{}_{l+2}\sigma_0 \cos(l+2)\theta. \end{aligned}\right\} \tag{34}$$

The restriction $l \leqslant 2$ in (34) is necessary since the neglected term of $O(\xi^2)$ in (32) gives rise to a term $3U_0 \cos\theta \xi_0^2 t^4$, which gives an extra contribution to ξ_3. The correct ξ_3 is then given by (34) plus $-\frac{3}{5}\,{}_1\sigma_0 \cos\theta\xi_0^2\,(= \frac{3}{5} U_0 \cos\theta \xi_0^2)$. Higher-order ξ_l terms must be similarly adapted.

As an example the case with $c(t)$ given by (31) with $\alpha = 0$ (or (29)) was evaluated in detail from (25) and (34) for $U_0 \neq 0$ and ξ_0, ξ_1, ξ_2 and ξ_3 were found. The bubble shape as a function of time is given by (19), with T and V defined in (5), where

$$\begin{aligned} \xi(\theta,t) = T^2\{ &-\tfrac{1}{2}V^2\cos 2\theta + TV[-\tfrac{1}{3}\cos 2\theta + \tfrac{1}{12}V^2(2\cos\theta - 2\cos 2\theta + 5\cos 3\theta)] \\ &+ T^2[-\tfrac{1}{12}\cos 2\theta + \tfrac{1}{24}V^2(5\cos\theta - 4\cos 2\theta + 11\cos 3\theta) \\ &+ \tfrac{1}{48}V^4(-11 + 2\cos\theta - 6\cos 2\theta + 11\cos 3\theta - 20\cos 4\theta)] \\ &+ T^3 V[\tfrac{1}{60}(5\cos\theta - 4\cos 2\theta + 11\cos 3\theta) + \tfrac{1}{240}V^2(-78 + 16\cos\theta \\ &- 53\cos 2\theta + 73\cos 3\theta - 144\cos 4\theta) \\ &+ \tfrac{1}{240}V^4(-32 + 88\cos\theta - 5\cos 2\theta + 50\cos 3\theta - 77\cos 4\theta + 110\cos 5\theta)] \\ &+ O(T^4)\}. \end{aligned} \tag{35}$$

In deriving this form the $O(T^5)$ term has been calculated including the contribution from the $O(\xi^2)$ term which is excluded in (32).

As mentioned above, the character of the change in bubble shape is the same if (30) is used to determine the $c(t)$ but considerably more algebra is involved. The $O(T^2)$ term, for example, in (35) becomes $-\frac{1}{3}V^2\cos 2\theta$.

When a bubble moves steadily in a fluidized bed with a velocity U_0, say, an approximate relation between U_0 and F is known to be $U_0^2F = 0{\cdot}25$ (see, for example, Murray 1965*b*), that is $V = 0{\cdot}5$. With $V = 0{\cdot}5$, figure 1 illustrates the changing bubble shape with time T. The actual time t' in seconds is found from (5).

When bubbles form naturally at the bottom of a bed the velocity of detachment is probably close to U_0 ($U_0 \doteqdot 0{\cdot}5/F^{\frac{1}{2}}$). Thus the case (35) and figure 1 is an approximation to this situation rather than the following case where $U_0 \equiv 0$.

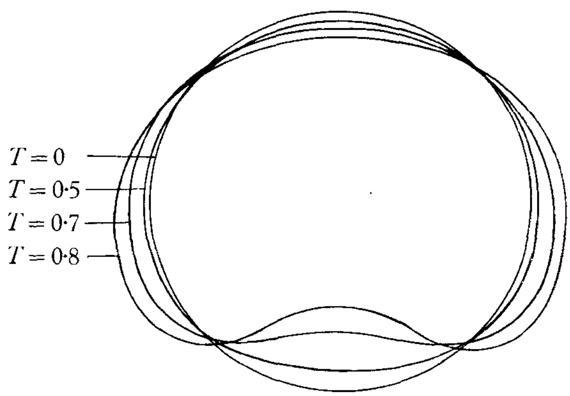

FIGURE 1. Bubble shape at various times: $U_0 = \frac{1}{4}F$.

The characteristic bubble shape, approximately that in figure 1 corresponding to $T = 0{\cdot}7$, $0{\cdot}8$, is evident in bubbles very close to the bottom of fluidized beds. From (5) this means that, with $T = 1$, $t' = (r_0/g)^{\frac{1}{2}}$. This implies, from (22), which gives $U(t)$, that the bubble has travelled a distance s, where ($T = 1$ implies $t = F^{\frac{1}{2}}$)

$$s/r_0 = \int_0^{F^{\frac{1}{2}}} U(t)\,dt = 1.$$

Thus the characteristic shape of a bubble in a fluidized bed is obtained by the time the bubble has travelled half its diameter. This estimate from the above analysis can only be an order-of-magnitude one in view of the different conditions which obtain at the bottom of a bed and those used above.

A measure of the rate of growth of the wake can be taken as $\dot{W}$, say, where $\dot{W}$ is the vertical velocity of the bottom of the bubble *relative* to the bubble centre.

Thus,
$$\dot{W} = -[\dot{\xi}]_{\theta=\pi}, \tag{36}$$
which from (35) gives

$$\dot{W} = T[V^2 + \tfrac{1}{4}TV(4+9V^2) + \tfrac{1}{6}T^2(2+20V^2+25V^4) + \tfrac{1}{24}T^3V(40+182V^2+181V^4) + O(T^4)]. \tag{37}$$

6. Results for a two-dimensional bubble starting from rest: $U_0 = 0$

The method, of course, is the same as above for $U_0 \neq 0$. Here we write

$$s_m = t^{2m-1} \sum_{l=0}^{\infty} {}_m\sigma_l\, t^l, \quad c(t) = \sum_{l=1}^{\infty} c_l\, t^l,\dagger \quad \xi = t^4 \sum_{l=0}^{\infty} \xi_l\, t^l, \tag{38}$$

where now, (21) and (32) give

$$ {}_m\sigma_l = [(m-1)/(2m-1+l)]\,[{}_{m-1}\sigma_l(c_1 - F^{-1}) + {}_{m-1}\sigma_{l-1}c_2 + \ldots + {}_{m-1}\sigma_1 c_l],$$
$$\xi_l = [1/(4+l)]\,[M_0\xi_{l-2} + \ldots + M_{l-2}\xi_0 - N_l] \quad (l \leqslant 6)\ddagger,$$

with

$$M_l = 2\cdot 3\,{}_2\sigma_{l-2}\cos 2\theta + 3\cdot 4\,{}_3\sigma_{l-4}\cos 3\theta + \ldots + \begin{cases} \tfrac{1}{4}(l+2)(l+4)\,{}_{\frac{1}{2}(l+2)}\sigma_0 \cos \tfrac{1}{2}(l+2)\,\theta, & (l \text{ even}) \\ +\tfrac{1}{4}(l+1)(l+3)\,{}_{\frac{1}{2}(l+1)}\sigma_1 \cos \tfrac{1}{2}(l+1)\,\theta, & (l \text{ odd}) \end{cases}$$

$$N_l = 2\,{}_2\sigma_l \cos 2\theta + 3\,{}_3\sigma_{l-2}\cos 3\theta + \ldots + \begin{cases} \tfrac{1}{2}(l+4)\,{}_{\frac{1}{2}(l+4)}\sigma_0 \cos \tfrac{1}{2}(l+4)\,\theta, & (l \text{ even}), \\ \tfrac{1}{2}(l+3)\,{}_{\frac{1}{2}(l+3)}\sigma_1 \cos \tfrac{1}{2}(l+3)\,\theta, & (l \text{ odd}). \end{cases}$$

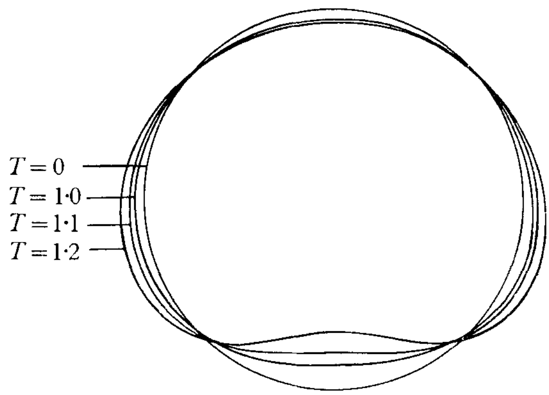

FIGURE 2. Bubble shape at various times: $U_0 = 0$.

The form for ξ from (39) was evaluated using (31) for general α to illustrate the effect of α, and to show that its value is not very crucial, and is given by

$$\begin{aligned}\xi(\theta, t) = T^4\{&-[1/6(\alpha+2)]\cos 2\theta + T^2[1/180(\alpha+2)^2] \\ &\times [5(\alpha+2)\cos\theta - 8(\alpha+1)\cos 2\theta + (5\alpha+22)\cos 3\theta] \\ &+ T^4[1/10{,}080(\alpha+2)^3][-35(\alpha+2)(\alpha+8) + 56(\alpha+1)(\alpha+2)\cos\theta \\ &- 2(83\alpha^2+206\alpha+200)\cos 2\theta + 8(7\alpha+62)(\alpha+1)\cos 3\theta \\ &- (35\alpha^2+434\alpha+1016)\cos 4\theta] + O(T^6)\}.\end{aligned} \tag{39}$$

The difference in the change in ξ for two different α, α_1, α_2, say, is approximately

$$\tfrac{1}{6}(\alpha_2 - \alpha_1)\,T^4/(\alpha_1+2)(\alpha_2+2) < T^4/64,$$

which is less than 2 % for allowable T. With the first term in (39) being $O(T^4)$ it implies, from (5), that the smaller the bubble the faster it will deform, within, of

† That $c_0 \equiv 0$ is indicated by all of (29)–(31).
‡ The lowest order in the $O(\xi^2)$ term in this case is $O(t^9)$.

course, the limitations of the various assumptions above. Figure 2 illustrates the bubble shape for various T, taking $\alpha = 1$ as an example.

The wake growth-rate characteristic $\dot{W}$ (from (36)) is given from (39) by

$$\dot{W} = T^3\left[\frac{2}{3(\alpha+2)}+\frac{(18\alpha+35)}{30(\alpha+2)^2}T^2+\frac{(92\alpha^2+479\alpha+646)}{280(\alpha+2)^3}T^4+O(T^6)\right]. \qquad (40)$$

It is clear from (37) and (40) (and figures 1 and 2) that the wake will grow much faster in circular bubbles initially moving with a given velocity, such as those occurring naturally at the bottom of a bed, than those starting from rest.

7. Three-dimensional case

The method used is the same as that given above and the results are given in the appendix for reference. In place of w_f, w_s, real scalar potentials are used with the appropriate harmonic function expansions being used. From (A 3) the first-order acceleration is twice that found in the two-dimensional case; that is, equivalent to an acceleration of $2g$.

Of particular interest is a comparison of the distortion (and hence an estimate of wake growth) rate. From (A 10) and (A 11) $\dot{W}$ is given by

$$\dot{W}_{U_0 \neq 0} = T[\tfrac{3}{2}V^2+\tfrac{3}{8}V(24+13V^2)T+O(T^2)], \qquad (41)$$

$$\dot{W}_{U_0=0} = T^3\left[\frac{4}{\alpha+2}+\frac{2(27\alpha+58)}{5(\alpha+2)^2}T^2+O(T^4)\right]. \qquad (42)$$

Equations (41), (42) correspond respectively to (37) and (40).

A comparison of (40) and (42) shows that the deformation from the circular or spherical state, and hence the wake growth, in the three-dimensional case is considerably faster than in the two-dimensional case: that is for bubbles starting from rest. When V has its approximate 'natural' values of 0·5, $1/\sqrt{3}$ in the two- and three-dimensional cases respectively the $O(T)$ terms in (37) and (41) are the same. The $O(T^2)$ term in (41), however, is larger than that in (37), and so even in this case the possible wake growth is again faster in the case of three-dimensional bubble motion. In view of the larger accelerations, however, the corresponding distance travelled in $t' \doteqdot (r_0/g)^{\frac{1}{2}}$ is here $r_0(1+1/\sqrt{3})$ as compared with r_0.

8. Conclusions

The constant-void-fraction solution and the irrotationality which results from it and the Oseen-type linearization allow solutions for small times to be found for the unsteady motion of initially circular and spherical bubbles in a fluidized bed. The solutions show that the characteristic shape of observed bubbles occurs in a time of $O[r_0/g)^{\frac{1}{2}}]$ from the bubbles' introduction into the bed. The accelerations in two and three dimensions are respectively g and $2g$ approximately.

Wakes appear to be present behind practically all bubbles and a function $\dot{W}$, which measures the velocity of the bottom of the bubble relative to the centre of the bubble, is given as a possible estimate of their growth.

The method and results given in this paper are valid only for small times and cannot be simply extrapolated to predict the final steady-state bubble motion prevalent in most gas-fluidized beds.

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Appendix: results for the initial motion of a three-dimensional spherical bubble

The method described in the two-dimensional case is directly applicable in this case. Instead of using complex variable forms for the perturbation quantities $\mathbf{v}'_f$, $\mathbf{v}'_s$ in (15), we use the scalar potentials ϕ_f, ϕ_s and write

$$\phi_f = \sum_{m=1}^{\infty} \frac{f_m P_m(\cos\theta)}{r^{m+1}}, \quad \phi_s = \sum_{m=1}^{\infty} \frac{s_m P_m(\cos\theta)}{r^{m+1}}, \tag{A 1}$$

where the P_m are Legendre polynomials of order m and from (14)

$$p' = -\frac{Z_0}{F} \sum_{m=1}^{\infty} \frac{(f_m - s_m) P_m(\cos\theta)}{r^{m+1}}. \tag{A 2}$$

Proceeding in exactly the same way as in §3 the solutions equivalent to (21), (22) are

$$\left.\begin{aligned} s_m &= -\left(\tfrac{1}{2}U_0 + \frac{1}{F}t\right) - m\int_0^t [U(\tau) - c(\tau)]\, s_{m-1}(\tau)\, d\tau, \\ f_m &= s_m - \delta_{m1}, \quad U(t) = -2s_1, \end{aligned}\right\} \tag{A 3}$$

and the equation for $\xi(\theta, t)$ is

$$\dot{\xi} - \xi \sum_{m=1}^{\infty} (m+1)(m+2) s_m P_m(\cos\theta) = -\sum_{m=2}^{\infty} (m+1) s_m P_m(\cos\theta) + O(\xi^2). \tag{A 4}$$

Equations (24), (25) hold exactly but with

$$\left.\begin{aligned} {}_m\sigma_l &= [m/(m+l-1)][{}_{m-1}\sigma_l(c_0 - U_0) + {}_{m-1}\sigma_{l-1}(c_1 - 2/F) \\ &\quad + {}_{m-1}\sigma_{l-2}\, c_2 + \dots + {}_{m-1}\sigma_0\, c_l], \\ \xi_l &= [1/(l+2)][\xi_{l-1} M_0 + \xi_{l-2} M_1 + \dots + \xi_0 M_{l-1} - N_l], \quad (l \leqslant 2) \\ M_l &= 2\cdot 3\, {}_1\sigma_l\, P_1 + 3\cdot 4\, {}_2\sigma_{l-1} P_2 + \dots + (l+2)(l+3)\, {}_{l+1}\sigma_0\, P_{l+1}, \\ N_l &= 3\, {}_2\sigma_l\, P_2 + 4\, {}_3\sigma_{l-1} P_3 + \dots + (l+3)\, {}_{l+2}\sigma_0\, P_{l+2}. \end{aligned}\right\} \tag{A 5}$$

Again, the $O(\xi^2)$ in (A 4) will contribute to ξ_3 with an extra $\frac{6}{5}U_0 P_1(\cos\theta)\,\xi_0^2$.

The evaluation of $c(t)$ can be similarly obtained and, for example, the equivalent of (30) gives in place of (33)

$$c(t) = -\left(\frac{\alpha+1}{\alpha+2}\right) \sum_{m=1}^{\infty} (m+1) s_m. \tag{A 6}$$

For the case $\alpha = 0$, the bubble shape is given by (19), (34) with ξ given by (equivalent of (35) and again with the $O(\xi^2)$ term included in $O(T^5)$ term)

$$\xi = T^2[A_0 + TA_1 + T^2A_2 + T^3A_3 + O(T^4)] \tag{A 7}$$

with

$$\begin{aligned}
A_0 &= -\tfrac{3}{4}V^2P^2,\\
A_1 &= V^3(\tfrac{3}{4}P_1P_2 - \tfrac{3}{8}P_2 + \tfrac{1}{2}P_3) - V3P_2,\\
A_2 &= V^4(-\tfrac{9}{16}P_1^2P_2 + \tfrac{9}{32}P_1P_2 - \tfrac{3}{8}P_1P_3 - \tfrac{9}{8}P_2^2 + \tfrac{3}{64}P_2 + \tfrac{9}{16}P_3 - \tfrac{5}{16}P_4)\\
&\quad + V^2(\tfrac{15}{8}P_1P_2 - \tfrac{3}{4}P_2 + P_3) - \tfrac{1}{2}P_2,\\
A_3 &= V^5(\tfrac{27}{80}P_1^3P_2 - \tfrac{27}{160}P_1^2P_2 + \tfrac{9}{40}P_1^2P_3 - \tfrac{9}{320}P_1P_2 - \tfrac{27}{80}P_1P_3 + \tfrac{3}{16}P_1P_4\\
&\quad + \tfrac{9}{4}P_1P_2^2 - \tfrac{9}{8}P_2^2 + \tfrac{69}{40}P_2P_3 + \tfrac{57}{320}P_2 + \tfrac{3}{32}P_3 - \tfrac{9}{16}P_4 + \tfrac{3}{16}P_5)\\
&\quad + V^3(\tfrac{9}{10}P_1P_2 - \tfrac{6}{5}P_1P_3 - \tfrac{81}{40}P_1^2P_2 - 3P_2^2 + \tfrac{21}{160}P_2 + \tfrac{57}{40}P_3 - \tfrac{7}{8}P_4)\\
&\quad + V(\tfrac{3}{2}P_1P_2 - \tfrac{3}{5}P_2 + \tfrac{4}{5}P_3).
\end{aligned}$$

The $U_0 = 0$ case has

$$\left.\begin{aligned}
s_m(t) &= t^{2m-1}\sum_{l=0}^{\infty} {}_m\sigma_l\, t^l,\\
{}_m\sigma_l &= [m/(2m+l-1)][{}_{m-1}\sigma_l(c_1 - 2/F) + {}_{m-1}\sigma_{l-1}c_2 + \ldots + {}_{m-1}\sigma_0 c_{l+1}],
\end{aligned}\right\} \tag{A 8}$$

with c_l given by combining (A 6) and (A 8). Tedious algebra gives, for ξ,

$$\xi = T^4[B_0 + T^2B_1 + T^4B_2 + O(T^6)] \tag{A 9}$$

with

$$\begin{aligned}
B_0 &= P_2/(\alpha+2),\\
B_1 &= [1/15(\alpha+2)^2][15(\alpha+2)P_1P_2 - 12(\alpha+1)P_2 + 16P_3],\\
B_2 &= [1/(140(\alpha+2)^3)][-280(\alpha+2)P_2^2 - 105(\alpha+2)^2P_1^2P_2\\
&\qquad + 84(\alpha+1)(\alpha+2)P_1P_2 - 112(\alpha+2)P_1P_3\\
&\qquad - 24(\alpha+1)(3\alpha-1)P_2 + 256(\alpha+1)P_3 - 160P_4].
\end{aligned}$$

In the three-dimensional case the approximate natural value of V is $\frac{1}{3}$. The measure of the wake growth comparable to (36) and (38) can be obtained from (A 7) and (A 9) by differentiation and setting $\theta = \pi$. Thus

$$\dot{W}_{U_0 \neq 0} = T[\tfrac{3}{2}U^2 + T\tfrac{1}{8}U(72 + 39U^2) + O(T^2)], \tag{A 10}$$

$$\dot{W}_{U_0 = 0} = T^3\left[\frac{4}{\alpha+2} + \frac{2(27\alpha+58)}{5(\alpha+2)^2}T^2 + O(T^4)\right]. \tag{A 11}$$

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